The Inverted Glass Harp

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We present an analytical treatment of the acoustics of liquid-filled wine glasses, or “glass harps”. The solution is generalized such that under certain assumptions it reduces to previous glass harp models, but also leads to a proposed musical instrument, the “inverted glass harp”, in which an empty glass is submerged in a liquid-filled basin. The versatility of the solution demonstrates that all glass harps are governed by a family of solutions to Laplace’s equation around a vibrating disk.

Tonal analysis of recordings for a sample glass are offered as confirmation of the scaling predictions.

Observant dinner guests have long been aware of the acoustic properties of wine glasses, specifically that a finger slid along the rim of the glass causes it to resonate, and that the resulting pitch depends on the amount of liquid in the glass. Wine glass musicians make use of this phenomenon by arranging several glasses filled to different heights. Entire songs can then be played on what has been called a “glass harp” (figure 1a). Excitement around glass harps has led to a handful of technical papers on wine glass acoustics. Building upon Rayleigh’s Theory of Sound [1], French [2] used energy conservation to develop the first mathematical model of a glass harp. Since then, researchers have confirmed the French model through experiment [3, 5, 6] and explored some details of the dynamics at the finger-glass interface [7, 8]. Some work has also been done to quantify the pitches that result from changing the distribution of water in and around the glass [3, 6].

It is on these themes that we propose a variation: the “inverted glass harp”. In an inverted glass harp, a single empty glass is held partially submerged in a basin of water, and the musician controls the pitch by changing the depth of the glass (figure 1b). While this new instrument can only play one pitch at a time, it has two key advantages over the original: (1) analog pitch control allows advanced techniques such as vibrato, trill, and glissando, and (2) the setup requires only one glass and no tuning in advance. A conceptual model for our analytical treatment of an inverted glass harp is shown in figure 2a. The glass is treated as a cylinder with radius $a$, height $h$, thickness $\epsilon$, and submergence depth $h_{\ell}$. The flow is assumed to be primarily 2-dimensional, with $z$-dependence entering only in the boundary conditions. This assumption will be later justified via experiment. By letting $a_{o}$ take on any value, we will treat several glass harp geometries at once. The case where $a_{o} \rightarrow \infty$ represents the idealized inverted glass harp, where the glass is in an infinite bath of fluid. When $a_{o} > a$ but finite, the wine glass is vibrating in the center of a larger vessel such as a cooking pot (figures 1b and 2b). When $a_{o} < a$, the fluid is contained within the walls of the glass, and the solution applies to a wine glass with a cylindrical rod placed in its center (figure 2c). The special case where $a_{o} = 0$ is the traditional glass harp problem solved by French. These geometries will offer a variety of experimental cases for which we can test our scaling predictions.

The motion of the glass is assumed to be the first radially symmetric harmonic for small amplitudes, that is, the displacement of the glass from its rest state $\delta = \delta_{0}\sin(\omega t)\cos(2\theta)\psi(z)$, where $\delta_{0}$ is the maximum dis...
placement at $z = h$, $\omega$ is the angular frequency of oscillation, $\psi(z)$ is the first eigenmode of a cantilevered beam, and $t$ is time. Holographic interferometry has shown that this harmonic dominates wine glass acoustics [5].

Because the decrease in pitch results from the added mass of the liquid, we will derive an expression for the specific kinetic energy of the fluid $\varepsilon$. To this end, we assume the flow to be incompressible and irrotational, define the velocity potential $u \equiv \nabla \phi$, and seek to solve the Laplace equation ($\nabla \cdot \nabla \phi = 0$) around a vibrating disk. After introducing dimensionless variables,

\[
R \equiv \frac{r}{a}, \quad T \equiv \omega t, \quad Z \equiv \frac{z}{h}, \quad H_t \equiv \frac{h_t}{h}, \quad \Psi \equiv \frac{\psi}{\delta_0},
\]

\[
\Delta \equiv \frac{\delta}{\delta_0}, \quad U \equiv \frac{u}{\omega a}, \quad \Phi \equiv \frac{\phi}{a \delta_0 \omega}, \quad E \equiv \frac{e}{\rho ha^2 \delta_0^2 \omega^2},
\]

where $u$ is flow speed and $\rho$ is the density of the liquid, the boundary conditions can be written as

1. $\frac{\partial \Phi}{\partial R}|_{R=R_o} = 0$ (no flux at the cylinder/basin walls),
2. $\frac{\partial \Phi}{\partial R}|_{R=1} = \frac{\partial \Delta}{\partial T}$ (matched velocity at the glass walls),
3. $\Phi|_{\theta=0} = \Phi|_{\theta=2\pi}$ (azimuthal periodicity), and
4. $|\Phi|$ bounded \{ $\forall R < 1$ (interior fluid) $\forall R > 1$ (exterior fluid) \}

Using separation of variables, the solution to Laplace’s equation for these boundary conditions can be written as

\[
\Phi(R, \theta, Z, T) = \cos(T) \cos(2\theta) \Psi(Z) \left( \frac{R^4 + R_0^4}{2R^2(1 - R_0^2)} \right),
\]

and the total dimensionless energy of the fluid $E_{KE,f}$ follows from integrating the specific energy over the volume of fluid:

\[
E_{KE,f} = \int_0^{H_t} \int_0^{2\pi} \int_0^R \frac{1}{2} (\nabla \Phi)^2 R \, dR \, d\theta \, dZ.
\]

With an expression for $E_{KE,f}^*$ in hand, we turn to the total energy budget of the system. French derives the following expressions for the kinetic and potential energies of an empty glass:

\[
E_{KE} = E_{KE,f}^* \sin^2 T, \quad E_{PE} = E_{PE,f}^* \cos^2 T,
\]

where

\[
E_{KE}^* = \frac{5\pi \rho_o \epsilon}{8 \rho a} \int_0^1 \Psi^2 dZ,
\]

\[
E_{PE}^* \approx \frac{3\pi}{8} \frac{\gamma_g e^3}{\rho_o^2 a^5} \left[ 1 + \frac{4}{3} \left( \frac{a}{h} \right)^4 \right] \int_0^1 \Psi^2 dZ,
\]

and $\rho_g$ and $\gamma_g$ are the density and elastic modulus of the glass. These expressions are unchanged for the inverted glass harp. Assuming ideal flow and a linearly elastic glass with no damping, the total energy of the system is conserved, that is

\[
E_{KE}^* + E_{KE,f}^* = E_{PE}^*.
\]

In the case of an empty glass, $E_{KE,f}^* = 0$, and solving equation 10 for $\omega$ gives the empty-glass resonant angular frequency $\omega_0$:

\[
\omega_0 = \left[ \frac{3 \gamma_g e^2}{5 \rho_o a^4} \left( 1 + \frac{4}{3} \left( \frac{a}{h} \right)^4 \right) \right]^{1/2}.
\]

When the glass is partially filled, all three terms in equation 10 are nonzero, and solving for $\omega$ gives a relation that includes boundary proximity $R_o$ and fill height $H_t$:

\[
\frac{\omega}{\omega_0} = \left( 1 + \frac{2M}{5} \frac{\rho a}{\rho_o} \left( \frac{R_0^4 + 1}{R_0^4 - 1} \right) \int_0^{H_t} \Psi^2 dZ \right)^{-\frac{1}{2}}.
\]

Linear beam theory predicts that $\Psi$ will be a superposition of hyperbolic functions. As French notes, however, the first eigenmode is well-estimated by the function $\Psi = Z^{3/2}$. To a good approximation then, we can estimate that $\omega$ follows the simpler relation

\[
\frac{\omega}{\omega_0} = \left( 1 + \frac{2M}{5} \frac{\rho a}{\rho_o} \left( \frac{R_0^4 + 1}{R_0^4 - 1} \right) H_t^2 \right)^{-\frac{1}{2}}.
\]

When $R_o = 0$, equation 13 reduces to a form similar to that derived by French, though by calculating the added mass exactly, our coefficients differ by a factor of 2. Interestingly, the case where $R_o \to \infty$ also reduces to the form derived by French. This equivalence is perhaps unsurprising since the conformal map $f(re^{i\theta}) = r^{-1}e^{i\theta}$ projects a disk into the external plane and preserves the satisfaction of Laplace’s equation. One effect of the equivalence is that an inverted glass harp should produce comparable pitches to a traditional glass harp. Indeed, our experiments will show that pitches for the internal and external cases are of the same order.
FIG. 3: Theoretical contour plots of dimensionless radial ($U_R$) and azimuthal ($U_\theta$) speeds for four different $R_o$ values. When $R_o < 1$, the wine glass is the outer circle; when $R_o > 1$, the wine glass is the inner circle.

For all other values of $R_o$, frequencies are lower than the $R_o = 0$ and $R_o \to \infty$ cases for the same $H_\ell$ values. This effect is best illustrated by considering contour plots of the radial and azimuthal speeds predicted by equation 2. As shown in figure 3, radial confinement (finite nonzero $R_o$) has little effect on the radial speeds, but causes significant increases in the azimuthal speeds near solid boundaries, as required by continuity. This increase in azimuthal speed raises the kinetic energy of the liquid phase, thereby increasing the effective mass and lowering the resonant frequency of the glass-liquid system.

Despite the mathematical equivalence of the internal and external cases, in practice the frequencies were found to be slightly higher for the external cases, presumably due to slight three-dimensional effects not included in this analysis. Because liquid outside the glass ($R_o > 1$) is forced by a convex surface and liquid inside the glass ($R_o < 1$) is forced by a concave surface, we expect the latter case to have more confinement-induced vertical motion and therefore a higher virtual mass coefficient. Indeed, the best-fit values for $M$ are 0.68 for $R_o > 1$ and 0.88 for $R_o < 1$, and their proximity to $O(1)$ indicates that the added mass dynamics are captured well.

To test the scaling predictions of equation 13, the resonant frequencies of a typical cylindrical wine glass ($\epsilon = 1.9$ mm, $a = 39$ mm, $h = 82$ mm, $\rho_g = 2470$ kg/m$^3$) were measured for a variety of $R_o$ values. For $R_o$ values less than 1, one of three cylindrical rods were placed in the center of the glass. For $R_o$ values greater than 1, the glass was partially submerged in one of three large cylindrical basins. To test the linearity of the added mass model, we considered one final case where the glass was filled both internally ($R_o = 0$) and externally ($R_o = 8.32$) to the same height. At each condition, the glass was struck by a wooden rod and the resulting sound was recorded with a cardioid microphone (Sennheiser E 835). While generated spectra are known to depend on the excitation method [7], the fundamental frequencies were found to be indistinguishable regardless of whether the glass was struck or rubbed with a wet finger. The discrete Fourier transform of the resulting sound wave (orange curve in figure 4) was smoothed with a low-pass filter (blue curve in figure 4), and the peak of the filtered signal was recorded as the resonant frequency $f$.

\[ f \equiv \log_{10}|F|^2, \] where $F$ is the magnitude of the Fourier transform of the sound wave. The sample signals shown are the raw (orange) and filtered (blue) transforms for the sample case where $H_\ell = 1$ and $R_o = 0.57$. The sample signals shown are the raw (orange) and filtered (blue) transforms for the sample case where $H_\ell = 1$ and $R_o = 0.57$.
To examine the importance of the liquid underneath the glass, an additional case was considered in the medium-sized basin ($R_o = 8.32$). In this test, silicon rubber was cured in the base of the basin such that water was only present above the stem of the glass. No observable difference was recorded in the resonant frequencies. It appears that the two-dimensionality of the flow causes the depth of the external basin to be largely irrelevant.

One final test was conducted in a small basin ($R_o = 2.0$). The frequencies at $H_t$ values near 1 were ≈ 5% higher than those predicted by equation 13. Presumably finite-amplitude or viscous effects become important at this extreme value of $R_o$, and the linear solution breaks down. Data from these two additional experiments will not be reported, but we mention them here for completeness.

The raw resonance data are given in table I. For reference, the note $F_5$ is ≈ 700 Hz and the note $D_6$ is ≈ 1175 Hz, so the musical range of the glass is approximately a major sixth. To plot the data, the height was rescaled based on equation 13, that is, using $M^{1/4}(R_o^2 + 1)/(R_o^2 - 1)^{1/4}H_t$ as the abscissa, where $M$ takes on one of two values, depending on whether the liquid is inside or outside the glass. Figure 5 shows the normalized resonant frequency as a function of this rescaled height. The collapse indicates that the model captures the essential physics of the problem. After determining only two virtual mass coefficients, the acoustic characteristics of the glass-liquid system can be predicted for a variety of both traditional and inverted glass harp setups.

The range and precision shown here, combined with the quick and easy setup, suggests that the inverted glass harp is a viable musical instrument. The pitches produced in the inverted case are slightly higher but similar to those in the traditional case, meaning that songs currently played on traditional glass harps could also be played on inverted harps. Interested musicians should note that the two largest basins were indistinguishable (the model predicts 99.9% similarity in tones once $R_o > 4.8$), and the basin height had no observable effect on the acoustics. Thus, so long as one uses a large enough basin, the acoustics of the harp will be reliably consistent. As shown in the case where liquid was internal and external, the internal fill height can be used to adjust the range of the instrument, similar to the way a capo adjusts the range of a guitar. The reliability of the tones makes the analog control of the inverted harp accessible to even the most amateur musicians. The authors, for example, were able to reproduce Mary Had a Little Lamb after only a few minutes of practice.

![Figure 5: Normalized resonant frequencies of the glass filled ($R_o < 1, M = 0.88$) or submerged ($R_o > 1, M = 0.68$) to various heights. The solid line shows the model (eq. 13, $M = 1$), while symbols show rescaled experimental data for $R_o = 0$ ($\bullet$), 0.20 ($\star$), 0.32 ($\circ$), 0.57 ($\blacklozenge$), 0.88 ($\blacktriangle$), 2.91 ($\blacklozenge$), 8.32 ($\star$), and 19.4 ($\diamond$). Black crosses show the case with both internal and external liquid ($R_o = 8.32, M = 0.68 + 0.88 = 1.56$).](image)

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